Appendix A: Semi-analytical Model for Resonator-enhanced Distributed Bragg Reflector

We introduce a semi-analytical method to calculate the transmittance and reflectance characteristics of a resonator-enhanced distributed Bragg reflector (RE-DBR). This method is computationally efficient and aids in optimizing design parameters for enhanced laser performance.

Consider a RE-DBR as shown in Fig. S1. The setup involves a microring resonator that is evanescently coupled to a bus waveguide, with a periodic array of grating posts arranged around part of the ring. Assuming both the bus waveguide and the ring resonator support a single transverse mode, the local optical field can be described using the complex amplitude of this mode. The relevant optical field amplitudes are defined in Fig. S1. The transmittance and reflectance of the resonator are given by $T = |t|^2 = |B_1/A_1|^2$ and $R = |r|^2 = |A_2/A_1|^2$. Our objective is to express t and r in terms of propagation constants, coupling strengths, and waveguide losses.

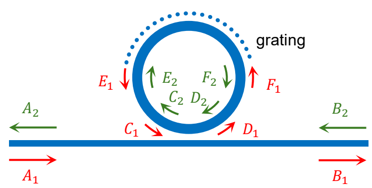

Figure S1: Schematic diagram of the RE-DBR. A Bragg grating partially covers a ring resonator, which is evanescently coupled to a bus waveguide.

1. Directional waveguide coupler

The coupling between the ring resonator and the bus waveguide is described by coupled mode theory [1]. According to this theory, the evolution of the optical field at the coupler depends on the waveguide coupling strength κ_c and the propagation constant mismatch $\delta = (\beta_2 - \beta_1)/2$, where β_1 and β_2 are the propagation constants for the bus waveguide and the ring resonator, respectively. Specifically, the input and output amplitudes of the ring resonator are related by the following equations:

$$\begin{aligned} A_2 &= \tau_1 B_2 + \tau_2 D_2 \\ B_1 &= \tau_1 A_1 + \tau_2 C_1 \\ C_2 &= -\tau_2^* B_2 + \tau_1^* D_2 \\ D_1 &= -\tau_2^* A_1 + \tau_1^* C_1, \end{aligned} \tag{S1}$$

where the self-coupling coefficient $\tau_1 = [\cos(ql_c) + j\frac{\delta}{q}\sin(ql_c))]e^{-j\delta l_c}$, the cross-coupling coefficient $\tau_2 = -j\frac{\kappa_c}{q}\sin(ql_c)e^{-j\delta l_c}$, and the effective coupling strength $q = \sqrt{\kappa_c^2 + \delta^2}$. These coupling coefficients can be determined through numerical simulations [2]. Since the role of propagation constant mismatch δ is considered, these equations apply to both symmetric and asymmetric couplers.

2. Bragg grating

The coupled mode theory for a Bragg grating is well established by modeling it as a one-dimensional periodic distribution of waveguide effective index [1, 3]. Assuming a grating period Λ, grating length l_g, coupling coefficient κ_g, and Bragg order N. We consider the grating to be lossless and mirror-symmetric. The amplitude transmission and reflection coefficients of the grating in terms of amplitude are:

$$\begin{aligned} t_g &= \frac{\rho_g}{\rho_g\cos(\rho_g l_g) + j\phi\sin(\rho_g l_g)}\exp(j\phi l_g) \\ r_g &= -\frac{j\kappa_g\sin(\rho_g l_g)}{\rho_g\cos(\rho_g l_g) + j\phi\sin(\rho_g l_g)}, \end{aligned} \tag{S2}$$

where $\phi = \beta_2 - N\pi/\Lambda$ and $\rho_g = \sqrt{\phi^2 - \kappa_g^2}$. The corresponding intensity transmission and reflection are given by $|t_g|^2$ and $|r_g|^2$. We proceed to derive the scattering matrix of the grating. Additional phase terms need to be added to t_g and r_g when formulating the scattering matrix. A detailed explanation is provided in the following paragraphs.

By definition, the grating scattering matrix S is a 2-by-2 matrix that satisfies

$$\begin{aligned} F_2 &= S_{11}F_1 + S_{12}E_2 \\ E_1 &= S_{21}F_1 + S_{22}E_2. \end{aligned} \tag{S3}$$

The values of the scattering matrix elements are restricted by the lossless condition, reciprocity, and mirror symmetry. These restrictions dictate that [4–6]

$$\begin{aligned} S^\dagger S &= I \\ S_{12} &= S_{21} \\ S_{11} &= S_{22}. \end{aligned} \tag{S4}$$

Moreover, since $|t_g|$ and $|r_g|$ represent the magnitudes of transmission and reflection coefficients, we have $|S_{11}| = |S_{22}| = |r_g|$ and $|S_{12}| = |S_{21}| = |t_g|$. Combining these equations, it can be shown that the scattering matrix must have the following form:

$$\begin{aligned} S_{11} &= S_{22} = -j|r_g|e^{-j\varphi} \\ S_{12} &= S_{21} = |t_g|e^{-j\varphi}, \end{aligned} \tag{S5}$$

where $\varphi \approx \beta_2 l_g$ corresponds to the phase difference in the optical field across the grating. The corresponding optical fields at the grating input and output ports are:

$$\begin{aligned} F_2 &= S_{11}F_1 + S_{12}E_2 = -j|r_g|e^{-j\varphi}F_1 + |t_g|e^{-j\varphi}E_2 \\ E_1 &= S_{21}F_1 + S_{22}E_2 = |t_g|e^{-j\varphi}F_1 - j|r_g|e^{-j\varphi}E_2. \end{aligned} \tag{S6}$$

Eq. S6 establishes the input-output relation for the Bragg grating.

Before concluding this section, we briefly introduce how to calculate the grating coupling coefficient κ_g. This coefficient can be derived from the longitudinal distribution of effective index $n_{\text{eff}}(z)$. When the effective index profile is a square wave with a duty cycle of 1/2, namely

$$n_{\text{eff}}(z) = \begin{cases} n_{\text{eff1}} & 0 < z < \dfrac{\Lambda}{2} \\ n_{\text{eff2}} & \dfrac{\Lambda}{2} < z < \Lambda \end{cases} \tag{S7}$$

the corresponding grating coupling coefficient can be expressed as [7, 8]

$$\kappa_g = \frac{\pi}{\Lambda}\frac{\Delta n_{\text{eff}}}{\bar{n}_{\text{eff}}}\frac{\sin(N\pi/2)}{N\pi}, \tag{S8}$$

where $\Delta n_{\text{eff}} = n_{\text{eff2}} - n_{\text{eff1}}$ and $\bar{n}_{\text{eff}} = (n_{\text{eff1}} + n_{\text{eff2}})/2$. Once the maximum and minimum effective indices over the grating are numerically simulated, then κ_g can be determined using Eq. S8.

3. Microring resonator

In the RE-DBR structure, the grating partially covers the ring resonator, and there are ring sections free of grating. Here we establish input-output relations for these grating-free ring sections. Since we have modeled the grating as a lossless component, the grating-free section of the ring resonator should account for all the optical loss within the ring for consistency of round-trip loss. The round-trip amplitude attenuation coefficient is $\alpha = \exp(-\rho l_r/2)$ with ρ

being the optical loss and l_r the ring circumference. Moreover, the phase shift across the grating-free section of the ring resonator is $\theta = \beta_2(l_r - l_g)$. To simplify the derivation, we assume the two ends of the grating are at the same distance from the resonator coupler. Our final results Eqs. S10 and S11 apply to general cases and do not depend on this assumption The optical fields in the grating-free sections of the ring resonator are then described by the following equations [1]:

$$
\begin{aligned}
E_2 &= C_2\sqrt{\alpha}\exp(-\frac{j\theta}{2}) = C_2\exp[-\frac{\rho l_r}{4} - \frac{j\beta_2(l_r-l_g)}{2}] \\
C_1 &= E_1\sqrt{\alpha}\exp(-\frac{j\theta}{2}) = E_1\exp[-\frac{\rho l_r}{4} - \frac{j\beta_2(l_r-l_g)}{2}] \\
D_2 &= F_2\sqrt{\alpha}\exp(-\frac{j\theta}{2}) = F_2\exp[-\frac{\rho l_r}{4} - \frac{j\beta_2(l_r-l_g)}{2}] \\
F_1 &= D_1\sqrt{\alpha}\exp(-\frac{j\theta}{2}) = D_1\exp[-\frac{\rho l_r}{4} - \frac{j\beta_2(l_r-l_g)}{2}].
\end{aligned} \tag{S9}
$$

4. Resonator-enhanced distributed Bragg reflector

Combining Eq. S1, Eq. S6, Eq. S9, and the implicit condition that $B_2 = 0$ (no light comes from the output port), we derive the expressions for the amplitude transmission and reflection coefficients for the RE-DBR:

$$
\begin{aligned}
t &= \tau_1 + \tau_2\tau_2^* t_0 \frac{\tau_1^* t_0 - |t_g|}{1 - 2\tau_1^*|t_g|t_0 + (\tau_1^*)^2 t_0^2} \\
r &= j\frac{\tau_2\tau_2^*|r_g|t_0}{1 - 2\tau_1^*|t_g|t_0 + (\tau_1^*)^2 t_0^2},
\end{aligned} \tag{S10}
$$

where $t_0 = \exp(-\rho l_r/2 - j\beta_2 l_r)$ is the roundtrip transmission coefficient for the ring resonator.

Eq. S10 provides an efficient method for calculating the transmittance and reflectance spectra of a RE-DBR. This approach eliminates the need for resource-intensive, full three-dimensional numerical simulations. Instead, it requires simulating the waveguide propagation constants and coupling strengths for the specific modes and wavelengths of interest. These simulated values can then be substituted into the equation to determine the transmission and reflection coefficients, which in turn yield the transmittance and reflectance spectra of the RE-DBR.

It is worth mentioning that a simplified version of Eq. S10 was proposed in [4], which applies when the ring resonator and the bus waveguide have their propagation constant matched. The formulation presented here is more general and is necessary to model RE-DBR with asymmetric directional coupler, as is the case in our experimental demonstration discussed in the text.

The effective cavity length of the RE-DBR is given by $L_{\text{eff}} = \frac{c}{2n_g}\frac{\partial\phi}{\partial\omega}$, where ϕ is the phase of the amplitude reflection coefficient $r = |r|e^{-j\phi}$, and n_g is the group index of the ring. Specifically, at the wavelength of maximum reflection, $\frac{\partial|r|}{\partial\omega} = 0$, it follows that $\frac{\partial r}{\partial\omega} = -j\frac{2n_g r}{c}L_{\text{eff}}$. By differentiating Eq. S10 with respect to frequency, we obtain an expression for the effective cavity length at the wavelength of maximum reflection:

$$
L_{\text{eff}} = \frac{1 - (\tau_1^* t_0)^2}{2[1 - 2\tau_1^*|t_g|t_0 + (\tau_1^* t_0)^2]} l_r. \tag{S11}
$$

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