Note on Numerical Calculation of Chirped Grating Spectra

Created on 2025-4-22
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Version 1

Introduction

In this note, we introduce how to numerically calculate the passive characteristics of a Bragg grating with a spatially varying period. Such a grating is usually referred to as a chirped grating and is known for its applications in dense wavelength division multiplexing (DWDM) communications and dispersion compensation for pulsed lasers. In particular, we will use the transfer matrix method to calculate the reflectance spectra and group delay of a linearly chirped single-mode grating. The notations used in this note follow the convention of [1]. The MATLAB script for the calculation is available via this link, which can reproduce the simulation results reported in [2] (Fig. 4(a)).

This note is organized as follows: First, we will briefly introduce the coupled mode theory for a no-chirp Bragg grating. Next, we will calculate the transfer matrix, reflection spectra, and group delay of a linearly chirped grating using the transfer matrix method. Finally, we will present the results of numerical experiments that reproduce figures from the literature.

Coupled Mode Theory for Bragg Grating

Consider a Bragg grating with the effective index distribution $n_{\text{eff}}(z) = n_0 + \delta n_{\text{eff}}(z)$, where $\delta n_{\text{eff}}(z)$ represents a perturbation to the uniform background effective index n_0 and is a periodic function of the longitudinal position z. Specifically, we assume the periodic perturbation to the effective index has the following form:

$$\delta n_{\text{eff}}(z) = \delta \bar{n}_{\text{eff}}[1 + v\cos(\frac{2\pi}{\Lambda}z)], \tag{1}$$

where $\delta \bar{n}_{\text{eff}}$ is the effective index perturbation averaged over periods (it is a constant in the context of this note), v represents the magnitude of effective index fluctuations, and Λ corresponds to the oscillation period.

The optical field evolution across the grating can be described by the coupled mode theory in an accurate and simply manner. It can be shown that, with a specific selection of reference system and under proper approximations, the amplitudes of the forward and backward light $R(z)$ and $S(z)$ satisfy a group of differential equations [1]:

$$\begin{aligned} \frac{dR}{dz} &= i\hat{\sigma}R(z) + i\kappa S(z) \\ \frac{dS}{dz} &= -i\hat{\sigma}S(z) - i\kappa^* R(z). \end{aligned} \tag{2}$$

The self-coupling coefficient $\hat{\sigma} = \delta + \sigma$ is defined by:

$$\begin{aligned} \delta &= 2\pi n_0(\frac{1}{\lambda} - \frac{1}{\lambda_D}) \\ \sigma &= \frac{2\pi}{\lambda}\delta \bar{n}_{\text{eff}}, \end{aligned} \tag{3}$$

where $\lambda_D = 2n_0\Lambda$ is the designed wavelength for Bragg scattering for an infinitesimal perturbation to effective index, i.e., $\delta n_{\text{eff}} \to 0$. The cross-coupling coefficient κ is defined by:

$$\kappa = \kappa^* = \frac{\pi}{\lambda} v \delta \bar{n}_{\text{eff}}. \tag{4}$$

Solving Eq. 2, we obtain the following relation that links optical field amplitudes at the grating input $R(0), S(0)$ and those at the grating output $R(L), S(L)$:

$$\begin{pmatrix} R(L) \\ S(L) \end{pmatrix} = \begin{pmatrix} \cosh(\gamma_B L) - i\frac{\hat{\sigma}}{\gamma_B}\sinh(\gamma_B L) & -i\frac{\kappa}{\gamma_B}\sinh(\gamma_B L) \\ i\frac{\kappa}{\gamma_B}\sinh(\gamma_B L) & \cosh(\gamma_B L) + i\frac{\hat{\sigma}}{\gamma_B}\sinh(\gamma_B L) \end{pmatrix} \begin{pmatrix} R(0) \\ S(0) \end{pmatrix}, \tag{5}$$

where $\gamma_B = \sqrt{\kappa^2 - \hat{\sigma}^2}$ and L is the length of the grating. The 2-by-2 matrix in Eq. 5, denoted by F, is the transfer matrix for the Bragg grating. The transfer matrix provides a complete description of the transmission and reflection characteristics of the grating and therefore concludes this section.

Transfer Matrix Method for Chirped Grating

So far, we have been discussing a Bragg grating where the effective index distribution is a rigorously periodic function, that is, the period at different positions is identical. In this section, we will focus on a Bragg grating where the period of index profile changes slowly and linearly over space. In this context, the effective index period is a linear function of the longitudinal position: $\Lambda(z) = \Lambda_0 + Cz$, where C is the chirp coefficient. We assume the longitudinal coordinate of the grating ranges from $-L/2$ to $L/2$, such that Λ_0 corresponds to the effective index period in the middle of the grating. The corresponding effective index distribution of the chirped grating is

$$\delta n_{\text{eff}}(z) = \delta \bar{n}_{\text{eff}}\{1 + v\cos[\frac{2\pi}{\Lambda_0}z + \phi(z)]\}, \tag{6}$$

where $\phi(z)$ represents the phase shift of the effective index profile due to the chirp. By definition, $\phi(z)$ satisfies the following condition:

$$\frac{d\phi}{dz} = \frac{2\pi}{\Lambda(z)} - \frac{2\pi}{\Lambda_0}. \tag{7}$$

To numerically calculate the transmission and reflection spectra of this chirped grating, we will employ the transfer matrix method. This involves segmenting the chirped grating into multiple sections and approximating each section as a no-chirp grating. By calculating the transfer matrix for these grating segments and multiplying them in a series, one can derive the total transfer matrix of the chirped grating. Below, we present a detailed mathematical formulation of this method.

Let's segment a linearly chirped grating into N segments. For the i-th segment, by recalling Eq. 3, the self-coupling coefficient $\hat{\sigma}_i = \delta_i + \sigma$ should follow the same form. That is,

$$
\begin{aligned}
\delta_i &= 2\pi n_0 (\frac{1}{\lambda} - \frac{1}{2n_0\Lambda(z_i)}) \\
&= 2\pi n_0 (\frac{1}{\lambda} - \frac{1}{2n_0\Lambda_0}) + 2\pi n_0 (\frac{1}{2n_0\Lambda_0} - \frac{1}{2n_0\Lambda(z_i)}) \\
&= 2\pi n_0 (\frac{1}{\lambda} - \frac{1}{\lambda_D(0)}) + 2\pi n_0 (\frac{1}{\lambda_D(0)} - \frac{1}{\lambda_D(z_i)}) \\
&= 2\pi n_0 (\frac{1}{\lambda} - \frac{1}{\lambda_D(0)}) + \frac{2\pi n_0}{\lambda_D(0)^2}\frac{d\lambda_D}{dz} z_i,
\end{aligned} \tag{8}
$$

where z_i is the longitudinal position at the middle of the i-th grating segment and $\lambda_D(z) = 2n_0\Lambda(z)$ is the designed wavelength for Bragg grating around the longitudinal position z.

The expression of σ is the same as in the case of a no-chirp grating and takes the same value for all grating segments:

$$
\sigma = \frac{2\pi}{\lambda}\delta\bar{n}_{\text{eff}}. \tag{9}
$$

Similarly, we can derive the expression of the cross-coupling coefficient κ by recalling Eq. 4,

$$
\kappa = \kappa^* = \frac{\pi}{\lambda}\upsilon\delta\bar{n}_{\text{eff}}. \tag{10}
$$

Note that κ takes the same value over all grating segments, so we do not need the subscript i for it. Let's denote $\gamma_{B,i} = \sqrt{\kappa^2 - \hat{\sigma}_i^2}$, the input-output relation for the i-th segment of the chirped grating follows:

$$
\begin{pmatrix} R(z_i + L_i/2) \\ S(z_i + L_i/2) \end{pmatrix} = \begin{pmatrix} \cosh(\gamma_{B,i}L_i) - i\frac{\hat{\sigma}_i}{\gamma_{B,i}}\sinh(\gamma_{B,i}L_i) & -i\frac{\kappa}{\gamma_{B,i}}\sinh(\gamma_{B,i}L_i) \\ i\frac{\kappa}{\gamma_{B,i}}\sinh(\gamma_{B,i}L_i) & \cosh(\gamma_{B,i}L_i) + i\frac{\hat{\sigma}_i}{\gamma_{B,i}}\sinh(\gamma_{B,i}L_i) \end{pmatrix} \begin{pmatrix} R(z_i - L_i/2) \\ S(z_i - L_i/2) \end{pmatrix}, \tag{11}
$$

where L_i is the length of the i-th grating segment, which satisfies $\sum_{i=1}^{N} L_i = L$. We denote the 2-by-2 transfer matrix in Eq. 11 as T_i. The transfer matrix of the entire chirped grating T can be calculated by multipying the transfer matrix for each grating segment $\{T_i\}$:

$$
T = T_N \times T_{N-1} \times ... \times T_1 = \Pi_{i=1}^{N} T_i. \tag{12}
$$

After obtaining the transfer matrix, the reflectivity R of the chirped grating and the group delay τ of the reflected light can be as follows:

$$
\begin{aligned}
r &= -\frac{T_{21}}{T_{22}} \\
R &= |r|^2 \\
\tau &= -\frac{d\arg(r)}{d\omega},
\end{aligned} \tag{13}
$$

where $\omega = 2\pi f$ is the angular frequency.

Numerical Experiments

Based on the theory described above, we developed a MATLAB code to numerically calculate the reflectivity and group delay for a linearly chirped grating. To demonstrate the correctness of our code, we use it to reproduce Fig. 4(a) in [2]. The parameters used in the calculation are listed below, and our results are consistent with those reported in the literature.

- Length of grating = 13.8 cm
- Grating period at the end: 528.14 nm
- Unperturbed effective index of the grating: 1.467
- Grating fringe visibility: 1
- Chirp coefficient: -2.46×10^{-8} m^{-1}
- Number of grating segments: 500
- Amplitude of effective index: 1×10^{-5}

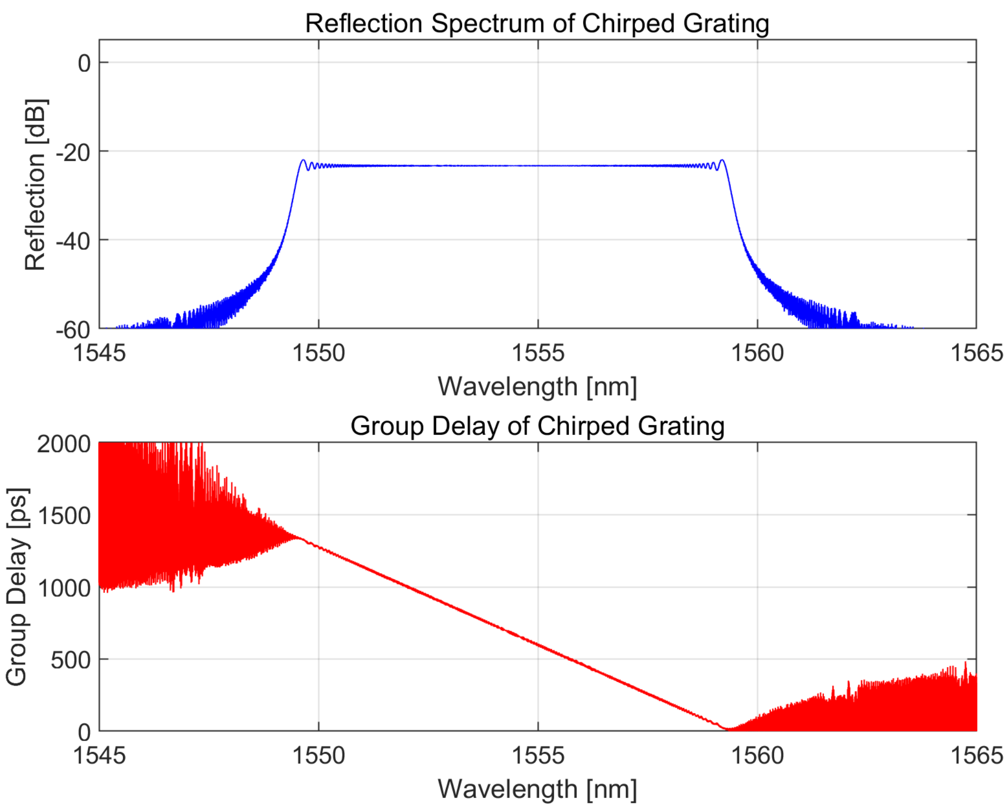

Figure 1: Simulated reflection spectrum and group delay of a linearly chirped grating.

Reference

1. Erdogan, Turan. "Fiber grating spectra." Journal of lightwave technology 15.8 (2002): 1277-1294.
2. Du, Zhenmin, et al. "Silicon nitride chirped spiral Bragg grating with large group delay." APL Photonics 5.10 (2020).