Note on Laser Linewidth Measurement

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Introduction

Lasers with narrow linewidths are essential for a variety of applications, such as optical atomic clocks, spectroscopy, and light detection and ranging (LIDAR). To accurately determine the linewidths of these lasers, it is necessary to measure their noise spectrum with high precision. In this note, we present a correlated self-heterodyne (COSH) setup capable of detecting frequency noise as low as 0.01 Hz2/Hz [1]. We provide a detailed explanation of the signal processing theory behind this scheme, along with annotations for the released code used to calculate the laser noise spectrum [2]. This setup meets the requirements for characterizing the linewidth of cutting-edge semiconductor lasers and facilitates the development of ultra-low-noise lasers.

Measurement Setup

The COSH setup is depicted in the figure below [1]. The optical output of the laser under test is split into two beams by a three-port acousto-optic modulator (AOM). One beam maintains the original laser frequency (unshifted), while the other undergoes a frequency shift of 55 MHz (shifted). The unshifted beam passes through a 1-km-long fiber delay line and is then combined with the shifted beam at a 50:50 fiber coupler, creating a modified Mach-Zehnder interferometer (MZI). Before the interference, the polarization of the shifted beam is adjusted to match that of the unshifted beam, maximizing the intensity of the beating note at the MZI output and optimizing the measurement sensitivity of the setup. The optical power difference between the two output ports of the modified MZI is detected by two balanced photodetectors (BPDs), each consuming half of the MZI's output power. The use of two BPDs is crucial for calculating cross-correlation and eliminating the contribution of independent BPD technical noise to the measured noise spectrum. The photocurrents are measured and recorded using a two-channel oscilloscope for further processing.

During data processing, the measured beating signal is subjected to a Hilbert transform (HT) and a fast Fourier transform (FFT). The Hilbert transform extracts the instantaneous phase of the beating signal, and the time-derivative of this instantaneous phase represents the instantaneous frequency. The Fourier transform of the instantaneous frequency yields the frequency noise power spectrum density (PSD) of the beating signal. Subsequently, the cross-correlation of the beating signals is calculated by multiplying the Fourier coefficients of the two channels. Finally, the cross-correlation of the beating signals is amplified by a processing gain $G(f)$, which compensates for the filtering effect of the modified MZI. This compensation suppresses the oscillating behaviors in the reconstructed noise spectrum, resulting in a more accurate representation of the laser's frequency noise characteristics.

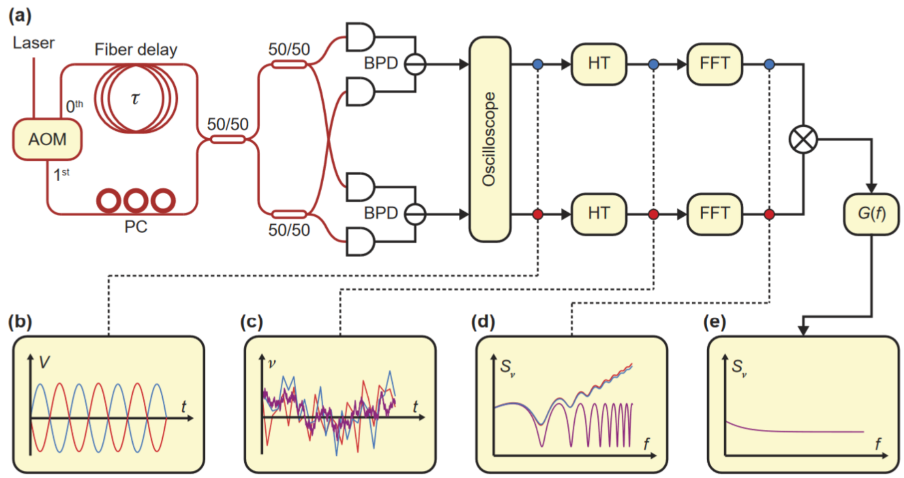

Figure 1: Cross-correlation self-heterodyne measurement setup [1].

Theoretical Derivation

Processing gain

In this section, we aim to determine the processing gain, which is a frequency-dependent signal gain that compensates for the modified MZI's frequency-dependent sensitivity when processing the beat signal. Mathematically, our goal is to find a factor $G(f)$ such that $S_\nu(f) = G(f)S_{\nu(\tau)}(f)$, where $S_\nu(f)$ and $S_{\nu(\tau)}(f)$ represent the frequency noise power spectral densities (PSDs) of the input laser and the beating signal, respectively. We will derive the relationship between the frequency noise spectra of the laser and beating signals, leading to an analytical expression for the processing gain.

Let $\phi(t)$ denote the instantaneous phase of the input laser, and $\nu(t) := \dot{\phi}(t)/2\pi$ be the corresponding instantaneous frequency. According to the Wiener-Khinchin theorem, the PSD and auto-correlation function of $\nu(t)$ form a Fourier-transform pair [1,3]:

$$S_\nu(f) = \int_{-\infty}^{\infty} \langle \nu(0)\nu(\tau)\rangle e^{2\pi i f\tau} d\tau. \tag{1}$$

Here, $\langle \nu(0)\nu(\tau)\rangle$ represents the ensemble average of $\nu(0)\nu(\tau)$ and is the auto-correlation function of $\nu(\tau)$. In the following section, we will establish a connection between the frequency noise PSD of the input laser and the beating output of the modified MZI.

In the measurement setup described in Section: Measurement Setup, the beating output of the modified MZI has an instantaneous frequency of $\nu(\tau) = \nu(t) - \nu(t-\tau) - f_c$, where $f_c = 55$ MHz represents the frequency of the AOM driving signal. It is crucial to note that $\nu(\tau)$ is the frequency of the photocurrent oscillation detected at the balanced photodetectors (BPDs), rather than the modulation frequency of the optical output of the modified MZI.

Applying the Wiener-Khinchin theorem, the frequency noise PSD of the beating signal can be expressed as:

$$S_{\nu(\tau)}(f) = \int_{-\infty}^{\infty} \langle [\nu(0) - \nu(-\tau) - f_c][\nu(\tau') - \nu(\tau'-\tau) - f_c]\rangle e^{2\pi i f\tau'} d\tau'. \tag{2}$$

In this equation, all terms proportional to f_c sum to zero because $\langle \nu(0) \rangle = \langle \nu(-\tau) \rangle = \langle (\tau') \rangle = \langle \nu(\tau' - \tau) \rangle$. Moreover, the term quadratically proportional to f_c^2 is $f_c^2 \delta(f)$, which contributes a DC component to the PSD of the beating signal and is not relevant to our noise spectrum analysis. By omitting these terms, Equation 2 can be simplified to a more concise form:

$$S_{\nu(\tau)}(f) = \int_{-\infty}^{\infty} \langle [\nu(0) - \nu(-\tau)][\nu(\tau') - \nu(\tau' - \tau)] \rangle e^{2\pi i f \tau'} d\tau'. \tag{3}$$

By employing the Wiener-Khinchin theorem and time translation invariance of ensemble averaging, we obtain:

$$\begin{aligned} S_{\nu(\tau)}(f) &= \int_{-\infty}^{\infty} \langle \nu(0)\nu(\tau') + \nu(-\tau)\nu(\tau' - \tau) - \nu(0)\nu(\tau' - \tau) - \nu(-\tau)\nu(\tau') \rangle e^{2\pi i f \tau'} d\tau' \\ &= 2S_\nu(f) - e^{2\pi i f \tau} \int_{-\infty}^{\infty} \langle \nu(0)\nu(\tau' - \tau) \rangle e^{2\pi i f(\tau' - \tau)} d\tau' - e^{-2\pi i f \tau} \int_{-\infty}^{\infty} \langle \nu(-\tau)\nu(\tau') \rangle e^{2\pi i f(\tau' + \tau)} d\tau' \\ &= (2 - \exp(2\pi i f \tau) - \exp(-2\pi i f \tau)) S_\nu(f) \\ &= 4 \sin^2(\pi f \tau) S_\nu(f). \end{aligned} \tag{4}$$

The equation indicates that the frequency noise PSD of the beating signal differs from that of the input laser by a modulation factor of $4 \sin^2(\pi f \tau)$. Consequently, once we have obtained the frequency noise PSD of the beating signal, we can recover the PSD of the input laser by applying an amplifying factor, which is the processing gain we are seeking. This processing gain can be expressed as:

$$G(f) = \frac{1}{4 \sin^2(\pi f \tau)}, \tag{5}$$

By multiplying the frequency noise PSD of the beating signal by this processing gain, we can accurately determine the frequency noise characteristics of the input laser, compensating for the modulation introduced by the modified MZI in the measurement setup.

Hilbert transform

The correlated self-heterodyne (COSH) scheme utilizes the Hilbert transform to extract the instantaneous phase of the beating signal. This section explains the underlying principles. Technically, only complex-valued signals have a well-defined phase, while real-valued signals need to be transformed into a complex-valued form, known as an analytical representation, to define their phase. The Hilbert transform plays a crucial role in constructing the analytical representation of a real-valued signal.

For a given real-valued signal $u(t)$, the Hilbert transform is defined as the convolution of $u(t)$ with the function $h(t) = 1/\pi t$ [4]:

$$H(u)(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{u(\tau)}{t - \tau} d\tau. \tag{6}$$

Using the Hilbert transform, we can construct a complex-valued signal $u_a(t) = u(t) + iH(u)(t)$. This complex-valued signal possesses two important properties:

1. The real part of $u_a(t)$ is $u(t)$;
2. $u_a(t)$ has no Fourier components with negative frequencies (proof can be found in [5]).

Due to these properties, $u_a(t)$ is called an analytic representation of the original real-valued signal $u(t)$. It can be shown that for any real-valued signal, there exists one and only one analytic representation. The imaginary part of this unique analytical representation is related to the real part through the Hilbert transform.

The instantaneous phase $\phi(t)$ of a real-valued signal $u(t)$ is defined as the phase of its analytic representation $u_a(t) = u(t) + iH(u)(t)$ [5]. Mathematically,

$$\phi(t) = \arg[u_a(t)] = \arctan[\frac{H(u)(t)}{u(t)}]. \tag{7}$$

This equation indicates that the instantaneous phase of a signal can be extracted based on its Hilbert transform, providing an integral tool for analyzing the noise spectrum of real-valued signals in the COSH scheme.

Gating and window function

In the data processing stage, the time series of instantaneous frequency $\nu(t)$, extracted using the Hilbert transform, is segmented before applying the Fourier transform. The first and last 40 ms of the time series are discarded due to signal distortion induced by the Hilbert transform [1]. The remaining data points are divided into non-overlapping segments, each with a time length of τ, which corresponds to the optical delay employed in the modified MZI. The finite length of signal τ is associated with a resolution bandwidth (RBW) of $1/\tau$. Frequency components in the laser noise spectrum with a spectral gap lower than the resolution bandwidth cannot be distinguished.

These signal segments are then processed to calculate the frequency noise PSD of the beating signal, following the typical windowing method [6]. It is important to note that segmenting should not be performed by a straightforward cutoff of the signal within a finite-length time window. Instead, the signal $\nu(t)$ is multiplied by a smooth *window function* $w(t)$ to extract a specified interval of the signal $\nu_{\text{gate}}(t)$. This process is known as *gating* and is mathematically expressed as:

$$\nu_{\text{gate}}(t) = w(t)\nu(t). \tag{8}$$

If the signal distortion induced by gating is negligible, the frequency noise PSD of the beating signal $S_{\nu(\tau)}(f)$ can be estimated from the Fourier coefficients of the gated signal $\hat{\nu}_{\text{gated}}(\tau, f)$. To this end, note that for the beating signal [7]:

$$S_{\nu(\tau)}(f) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\nu(\tau, t)|^2 dt. \tag{9}$$

Assuming that the windowing function $w(t)$ is nearly unity within a time frame $[-T_0/2, T_0/2]$ and close to zero outside this frame, we have $T_0 \approx \int_{-\infty}^{\infty} w(t)^2 dt$. It follows that:

$$
\begin{aligned}
S_{\nu(\tau)}(f) &\approx \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |\nu(\tau,t)|^2 dt \\
&\approx \frac{1}{\int_{-\infty}^{\infty} w(t)^2}\int_{-T_0/2}^{T_0/2} |\nu(\tau,t)|^2 w(t)^2 dt \\
&\approx \frac{1}{\int_{-\infty}^{\infty} w(t)^2}\int_{-\infty}^{\infty} |\nu(\tau,t)|^2 w(t)^2 dt \\
&= \frac{1}{\int_{-\infty}^{\infty} w(t)^2}\int_{-\infty}^{\infty} |\nu_{\text{gated}}(\tau,t)|^2 dt \\
&= \frac{\int_{-\infty}^{\infty} |\hat{\nu}_{\text{gated}}(\tau,f)|^2 df}{\int_{-\infty}^{\infty} w(t)^2 dt},
\end{aligned}
\tag{10}
$$

where the last equation is derived from Parseval's theorem [8], and $\hat{\nu}_{\text{gated}}(\tau, f)$ is defined as:

$$
\hat{\nu}_{\text{gated}}(\tau,f) = \int_{-\infty}^{\infty} \nu(\tau,t) w(t) \exp(2\pi i f t) dt. \tag{11}
$$

Equation 10 allows for estimating the frequency noise PSD of the beating signal from the Fourier coefficients of the segmented signals, providing a practical approach to characterize the noise properties of the laser under test.

Cross-correlation function

In this section, we explain (1) how to calculate the cross-correlation function based on measurement outcomes and (2) why the calculated cross-correlation function can serve as an estimate of the frequency noise PSD of the beating signal. For simplicity, let's consider an infinitely long signal without segmenting.

We assume that the instantaneous phases of the beating signals detected at the two BPDs are:

$$
\begin{aligned}
\phi_1(t) &= \phi(t) + \phi_{\text{BPD},1}(t), \\
\phi_2(t) &= \phi(t) + \phi_{\text{BPD},2}(t),
\end{aligned}
\tag{12}
$$

where ϕ is the instantaneous phase of the photocurrent at the BPDs without technical noise, and $\phi_{\text{BPD},1/2}$ denotes the contribution of technical BPD noise. The phase signal can be transformed into a frequency signal by:

$$
\begin{aligned}
\nu_1(\tau,t) &= \frac{\dot{\phi}_1(t)}{2\pi} = \nu(\tau,t) + \nu_{\text{BPD},1}(t), \\
\nu_2(\tau,t) &= \frac{\dot{\phi}_2(t)}{2\pi} = \nu(\tau,t) + \nu_{\text{BPD},2}(t).
\end{aligned}
\tag{13}
$$

The cross-correlation function of these two frequency signals, expressed in the frequency domain, is:

$$
r_{12}(\tau,f) = \int_{-\infty}^{\infty} \langle \nu_1(\tau,0)\nu_2(\tau,t)\rangle \exp(2\pi i f t) dt. \tag{14}
$$

Assuming that $\nu(\tau, t)$, $\nu_{\mathrm{BPD},1}$, and $\nu_{\mathrm{BPD},2}$ are independent random variables and $\langle \nu_{\mathrm{BPD},1/2} \rangle = 0$, we have:

$$r_{12}(\tau, f) = \int_{-\infty}^{\infty} \langle \nu(\tau, 0)\nu(\tau, t) \rangle \exp(2\pi i f t) dt = S_{\nu(\tau)}(f). \tag{15}$$

In the last equation, we have employed the Wiener-Khinchin theorem. Equation 15 indicates that the frequency noise PSD of the beating signal can be calculated as the cross-correlation function of the frequency noise measured at the two oscilloscope channels. The contribution of independent technical BPD noise is eliminated when performing ensemble averaging for the cross-correlation function, demonstrating the robustness of the COSH scheme against technical PD noise.

Next, we establish a relation between the cross-correlation function $r_{12}(\tau, f)$ and the product of the Fourier coefficients evaluated at the two channels. Note that the ensemble average of an infinitely long random signal with time-translation-invariant statistics at a certain moment is equal to its average value across the infinite time frame. Therefore, we can transform Eq. 14 to:

$$\begin{aligned} r_{12}(\tau, f) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu_1(\tau, t')\nu_2(\tau, t + t') \exp(2\pi i f t) dt dt' \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu_1(\tau, t')\nu_2(\tau, t') \exp(2\pi i f (t - t')) dt dt' \ (t + t' \to t) \\ &= \hat{\nu}_1(\tau, f)\hat{\nu}_2^*(\tau, f). \end{aligned} \tag{16}$$

This equation suggests that the cross-correlation function can be calculated as the product of the Fourier coefficients of the frequency noise measured at the two oscilloscope channels. Equation 16 can be readily adapted for finite-length segmented signals $\nu_{1/2,\mathrm{gated}}(t) = w(t)\nu_{1/2}(t)$ as:

$$r_{12,\mathrm{gated}}(\tau, f) \approx \frac{\hat{\nu}_1(\tau, f)\hat{\nu}_2^*(\tau, f)}{\int_{-\infty}^{\infty} w(\tau')^2 d\tau'},$$

where the denominator on the right-hand side of the equation comes from the approximation of the ensemble average of a signal at a given moment by the average value within a finite-length time frame.

Reference

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