Note on Simulation of Thermo-refractive Noise in Silicon Nitride Resonators

Created on Apr. 26, 2025
Author Di Yu (yudi.0211@foxmail.com)

Introduction

Small-sized objects experience random temperature fluctuations due to energy exchange with the environment, known as thermal fluctuations. In optical microresonators, these fluctuations lead to variations in the resonant frequency due to the thermo-optic effect. These variations in resonant frequency are known as thermo-refractive noise (TRN).

TRN is a significant source of noise in narrow-linewidth semiconductor lasers. It is often the predominant noise source for offset frequencies between 10 kHz and 100 kHz. Furthermore, TRN presents a challenge for achieving sub-Hz linewidths in integrated lasers at room temperature.

In this note, I will explain how to numerically simulate thermo-refractive noise in a silicon nitride (SiN) microresonator using COMSOL. The simulation file, developed based on the methodologies outlined in references [1] and [2], can be accessed via this link.

Fluctuation-Dissipation Theorem

The calculation of thermo-refractive noise relies on the fluctuation-dissipation theorem. This theorem states that any process dissipating energy by converting it into heat has a corresponding reverse process related to thermal fluctuations. For instance, when a ball falls into water, its speed decreases due to water friction, which transforms the ball's kinetic energy into heat. Simultaneously, water molecules collide with the ball, causing it to oscillate randomly. This oscillation represents thermal fluctuations and is the reverse process of the energy dissipation caused by water friction.

Mathematically, the fluctuation-dissipation theorem can be expressed as follows: Consider a system with a generalized force f acting on a generalized coordinate x. If the system is linear, the temporal response function $\alpha(t)$ is defined by:

$$x(t) = \int \alpha(\tau) f(t-\tau) d\tau. \tag{1}$$

The spectral density of thermal fluctuations in x is given by:

$$S_x(\omega) = 2\hbar \Im[\alpha(\omega)] \coth(\frac{\hbar\omega}{2k_B T}), \tag{2}$$

where $\Im[\alpha(\omega)]$ is the imaginary part of the Fourier transform of $\alpha(t)$. This term represents the system's dissipation rate. Hence, the equation above connects the magnitude of fluctuations to the dissipation rate, which is the essence of the fluctuation-dissipation theorem.

The fluctuation spectral density of a physical quantity y, which depends on x, can be derived similarly. To prepare for calculating thermo-refractive noise, we focus on the spectral density of thermal fluctuations in $y = \int x(\mathbf{r}, t) q(\mathbf{r}, t) d^3\mathbf{r}$. It can be shown that [3], given a probe force $f = F_0 \cos(\omega t) q(\mathbf{r})$ acting on the system, the energy dissipation over a force period $2\pi/\omega$ is proportional to the thermal fluctuation spectral density of $y(\omega)$:

$$S_y(\omega) = 2\hbar \frac{W_{\text{diss}}}{\pi F_0^2} \coth(\frac{\hbar\omega}{2k_B T}). \tag{3}$$

To ensure this equation holds, the probe force f must be energy-conjugate to x; in other words, fdx or xdf should correspond to heat dQ or work dW. For example, temperature T and entropy S form an energy-conjugate pair because $dQ = TdS$. This energy-conjugate relationship is crucial for our subsequent discussion.

Thermal Fluctuations in Optical Cavity

In the context of thermal fluctuations within an optical cavity, the fluctuation-dissipation theorem can be expressed as follows: the thermo-refractive noise in the cavity is related to the dissipation rate of optical field energy. This relationship is formulated by [1,2]:

$$S_{\delta f/f}(\omega) = 2\hbar \frac{W_{\text{diss}}}{\pi F_0^2} \coth(\frac{\hbar\omega}{2k_B T}), \tag{4}$$

where $S_{\delta f/f}(\omega)$ represents the single-sided spectral density of the normalized fluctuation in the cavity's resonant frequency $\delta f/f$, W_{diss} is the energy dissipated over one period of the thermal fluctuation (namely $2\pi/\omega$), F_0 is a reference energy set to 1 J, $\hbar$ is the reduced Planck constant, k_B is the Boltzmann constant, and T is the environmental temperature.

The dissipated energy W_{diss} is determined by solving the heat transfer equation with the optical field acting as the heat source:

$$\rho C_V \delta\dot{T} - \kappa \nabla^2(\delta T) = \dot{Q} = T\dot{S}. \tag{5}$$

In this equation, ρ represents the material density, C_V is the specific heat capacity, δT is the temperature variation, κ denotes thermal conductivity, Q is heat, T is the environmental temperature, and S signifies entropy. To convert this differential equation from the time domain to the frequency domain, we substitute $\delta T(t) = \Re(\tilde{T}(\omega)e^{-i\omega t})$ and $S(t) = \Re(\tilde{S}(\omega)e^{-i\omega t})$ into Eq. 2, leading to

$$i\omega\rho C_V \tilde{T} + \kappa\nabla^2\tilde{T} = i\omega T\tilde{S}. \tag{6}$$

Once the ampplitude of temperature fluctuation $\tilde{T}$ is determined, the dissipated energy can be calculated as follows:

$$W_{\text{diss}} = \int \frac{\kappa}{T}(\nabla\delta T)^2 d^3\mathbf{r}dt = \int \frac{\pi\kappa}{\omega T}|\nabla\tilde{T}|^2 d^3\mathbf{r}. \tag{7}$$

Before solving Eq. 7, we need to know the expression of $\tilde{S}$. Since $\tilde{S}$ is energy-conjugate to the temperature δT, by the fluctuation-dissipation theorem, we only need to determine the relationship between δT and the physical quantity of our interest, namely the resonant frequency variation δf. According to the perturbation theory of the electromagnetic field [4]:

$$\frac{\delta f}{f} = -\frac{1}{2}\frac{\int \Delta\epsilon|\mathbf{E}|^2 d^3\mathbf{r}}{\int \epsilon|\mathbf{E}|^2 d^3\mathbf{r}} = -\frac{\int \epsilon_0\sqrt{\epsilon_r}\beta\delta T|\mathbf{E}|^2 d^3\mathbf{r}}{W^{\mathrm{WGM}}}. \tag{8}$$

Here, ϵ_0 is the permittivity of vacuum, ϵ_r is the relative permittivity of the dielectric, β is the thermo-optic coefficient, W^{WGM} is the normalized factor equal to four times the time-averaged electric field energy:

$$W^{\mathrm{WGM}} = \int \epsilon_0\epsilon_r|\mathbf{E}|^2 dV. \tag{9}$$

Applying the fluctuation-dissipation theorem (with $x = \delta T$, $f = S$, $y = \delta f/f$, and $q = -\epsilon_0\sqrt{\epsilon_r}\beta|\mathbf{E}|^2/W^{\mathrm{WGM}}$), the complex amplitude of entropy $\tilde{S}$ takes the form:

$$\tilde{S} = -F_0\epsilon_0\sqrt{\epsilon_r}\beta|\mathbf{E}|^2/W^{\mathrm{WGM}}. \tag{10}$$

In summary, the process of calculating thermo-refractive noise involves the following steps:

1. Simulate the electromagnetic field: Begin by simulating the distribution of electromagnetic field for the mord of interest, denoted as $\mathbf{E}(\mathbf{r}, \omega)$.
2. Calculate heat dissipation: Use the expression for $\tilde{S}(\mathbf{r}, \omega)$ (as given in Eq. 10) to determine the corresponding heat dissipation.
3. Solve the heat transfer equation: Solve the heat transfer equation (Eq. 5) to find the temperature fluctuation, $\delta T(\mathbf{r}, \omega)$.
4. Calculate dissipated energy: Compute the dissipated energy, W_{diss}, using Eq. 7.
5. Apply the fluctuation-dissipation theorem: Use the fluctuation-dissipation theorem (Eq. 4) to calculate $S_{\delta/f}$.
6. Determine frequency noise spectral density: Once $S_{\delta f/f}$ is obtained, derive the single-sideband frequency noise spectral density using the formula: $S_{\delta f}(\omega) = f^2 S_{\delta f/f}(\omega)/2$. Note that f is the eigen-frequency of the optical cavity mode.

Simulation of Thermo-refractive Noise Using Finite Element Analysis

---

To simulate thermo-refractive noise in a silicon nitride ring resonator, we implemented finite element analysis. The silicon nitride waveguide is characterized by a width of 4.6 µm and a thickness of 100 nm. It is encapsulated by a 2-um-thick cladding oxide, with air above and an 8-um-thick buried oxide below. The ring has a radius of 1 mm, corresponding to a resonator free spectral range (FSR) of 30 GHz. The simulation parameters for thermo-refractive noise are as follows:

- Resonant wavelength of the optical cavity mode: 1550 nm
- Environmental tempearture: 25 degC
- Refractive index of silicon nitride: 1.996
- Refractive index of silicon dioxide: 1.444
- Thermo-optic coefficient of silicon nitride: 2.45E-5 K^{-1}
- Thermo-optic coefficient of silicon dioxide: 8.53E-6 K^{-1}
- Specific heat capacity of silicon nitride: 800 J/(Kg·K)
- Specific heat capacity of silicon dioxide: 705 J/(Kg·K)
- Specific heat capacity of air: 1012 J/(Kg·K)
- Density of silicon nitride: 3.29E+3 $\mathrm{kg/m^3}$
- Density of silicon dioxide: 2.196E+3 $\mathrm{kg/m^3}$
- Density of air: 1.293 $\mathrm{Kg/m^3}$

- Thermal conductivity of silicon nitride: 30 W/(m·K)
- Thermal conductivity of silicon dioxide: 1.38 W/(m·K)
- Thermal conductivity of air: 0.024 W/(m·K)

Using these parameters, we simulated thermo-refractive noise with COMSOL. We then compared the simulation results to the experimentally measured frequency noise spectrum of a hybrid integrated laser that has an external cavity of similar structure. The comparison is illustrated below:

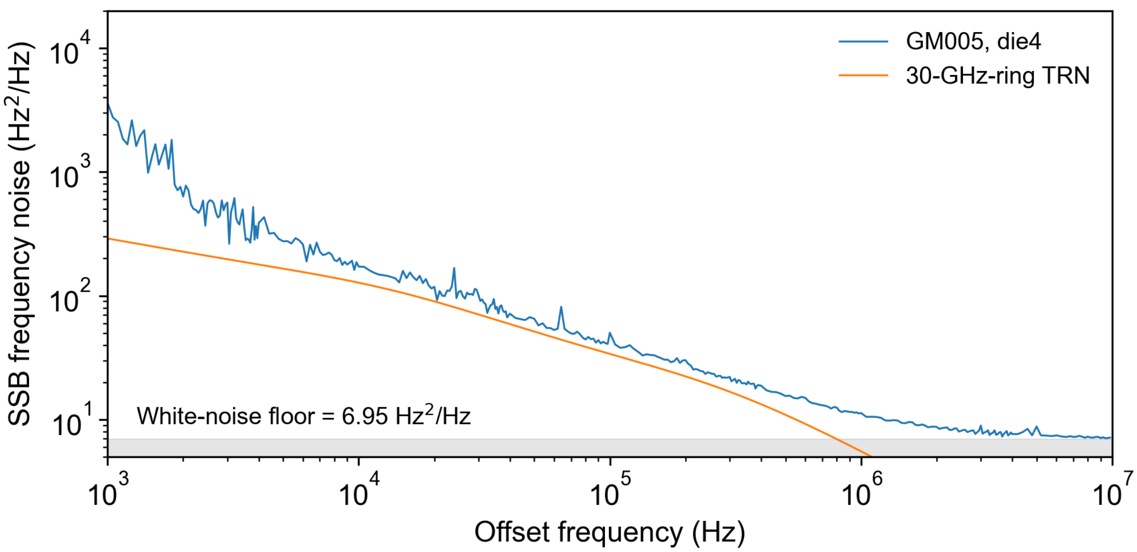

Figure 1: Comparison of simulated TRN and experimentally measured laser frequency noise.

The simulated thermo-refractive noise aligns well with the measured laser frequency noise at offset frequencies between 10 kHz and 100 kHz, confirming the accuracy of our simulation.

Reference

1. Huang, Guanhao, et al. "Thermorefractive noise in silicon-nitride microresonators." Physical Review A 99.6 (2019): 061801.
2. Kondratiev, N. M., and M. L. Gorodetsky. "Thermorefractive noise in whispering gallery mode microresonators: Analytical results and numerical simulation." Physics Letters A 382.33 (2018): 2265-2268.
3. Levin, Yu. "Internal thermal noise in the LIGO test masses: A direct approach." Physical Review D 57.2 (1998): 659.
4. Meade, Robert D. VE, Steven G. Johnson, and Joshua N. Winn. "Photonic crystals: Molding the flow of light." (2008).