Note on Waveguide Optics

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Introduction

In this note, we have presented a self-contained formulation of waveguide optics. Beginning with the definition of waveguide modes, we derived the master equation governing their transverse field distributions and showed that the problem of finding guided modes is equivalent to determining the eigenvectors of a Hermitian operator—an analogy that closely parallels the formalism of quantum mechanics.

We then introduced an inner product for guided electromagnetic fields and demonstrated that modes associated with different propagation constants are orthogonal under this definition. Building on this, we constructed an orthonormal basis of modal fields, which greatly simplifies the representation of arbitrary guided fields and their associated power flow.

Finally, we derived a practical expression for the power coupling efficiency at the interface between two waveguides, showing that it can be expressed compactly in terms of the mode overlap between the incident field and the target guided mode. This expression is widely used in numerical mode solvers and provides essential insight into mode conversion and coupling phenomena in integrated photonics.

Basics of Waveguide Optics

Derivation of master equation

Electromagnetic (EM) fields in a linear medium can always be written as superpositions of time-harmonic fields. A single time-harmonic component takes the form

$$\mathbf{E} = \mathbf{E}(x,y,z)\exp(-i\omega t), \mathbf{H} = \mathbf{H}(x,y,z)\exp(-i\omega t).$$

In the following, $\mathbf{E}$ and $\mathbf{H}$ refer exclusively to the **spatial field amplitudes** $\mathbf{E}(x,y,z)$ and $\mathbf{H}(x,y,z)$.

For these amplitudes, Maxwell's equations become

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mu_0\mathbf{H} \\ \nabla \times \mathbf{H} &= -i\epsilon\omega\mathbf{E} \\ \nabla \cdot (\epsilon\mathbf{E}) &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \end{aligned} \tag{1a-d}$$

Taking the curl of the first two equations yields

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \omega^2\mu_0\epsilon\mathbf{E} \\ \nabla \times (\frac{1}{\epsilon}\nabla \times \mathbf{H}) &= \omega^2\mu_0\mathbf{H} \end{aligned}$$

Using the vector identities (f is a scalar field and $\mathbf{v}$ denotes a vector field)

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$
$$\nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f \nabla \times \mathbf{v},$$

The curl equations may be recast as

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= \omega^2 \mu_0 \epsilon \mathbf{E} \\ -\frac{1}{\epsilon^2} \nabla \epsilon \times (\nabla \times \mathbf{H}) + \frac{1}{\epsilon}(\nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}) &= \omega^2 \mu_0 \mathbf{H} \end{aligned} \tag{2a-b}$$

Consider now a dielectric waveguide whose permittivity is invariant along the z-axis but may vary in the transverse xy-plane: $\epsilon = \epsilon(x, y)$. Guided fields can be written as modal fields with a longitudinal phase factor:

$$\mathbf{E}(x, y, z) = \mathbf{e}(x, y) \exp(i\beta z), \mathbf{H}(x, y, z) = \mathbf{h}(x, y) \exp(i\beta z), \tag{3}$$

Here, $\mathbf{e}(x, y)$ and $\mathbf{h}(x, y)$ describe the transverse field distribution, and β is the propagation constant.

Substituting Eq. 3 into the divergence condition $\nabla \cdot (\epsilon \mathbf{E})$ gives

$$\nabla_t \cdot (\epsilon \mathbf{e}_t) + i\beta \epsilon e_z = 0, \tag{4}$$

where $\nabla_t = \hat{x}\partial_x + \hat{y}\partial_y$ and $\mathbf{e}_t = e_x \hat{x} + e_y \hat{y}$. $\hat{x}$ and $\hat{y}$ represent the unit vectors along x- and y-axis, respectively.

Eq. 4, together with Maxwell's equation Eq. 1a, implies that the full 3D field is completely determined once $\mathbf{e}_t(x, y)$ and β are known. In the following, our goal is to find an equation for $\mathbf{e}(x, y)$ and β.

Substituting Eq. 3 into Eq. 2a, the left side of the equation becomes:

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= \nabla(\nabla \cdot (\mathbf{e} \exp(i\beta z))) - \nabla^2(\mathbf{e} \exp(i\beta z)) \\ &= \nabla(\nabla_t \cdot \mathbf{e} \exp(i\beta z) + i\beta e_z \exp(i\beta z)) - \nabla_t^2 \mathbf{e} \exp(i\beta z) + \beta^2 \mathbf{e} \exp(i\beta z) \\ &= (\nabla_t + i\beta \hat{z})(\nabla_t \cdot \mathbf{e} + i\beta e_z) \exp(i\beta z) - \nabla_t^2 \mathbf{e} \exp(i\beta z) + \beta^2 \mathbf{e} \exp(i\beta z) \end{aligned}$$

The right side of the equation is:

$$\omega^2 \mu_0 \epsilon \mathbf{E} = \omega^2 \mu_0 \epsilon \mathbf{e} \exp(i\beta z).$$

Therefore we arrive in the following equation:

$$(\nabla_t + i\beta \hat{z})(\nabla_t \cdot \mathbf{e} + i\beta e_z) - \nabla_t^2 \mathbf{e} = (\omega^2 \mu_0 \epsilon - \beta^2)\mathbf{e}. \tag{5}$$

To solve for $\mathbf{e}_t$, let's take the transverse component of Eq. 5 and use Eq. 4 to eliminate e_z, yielding

$$\nabla_t(\nabla_t \cdot \mathbf{e} - \frac{1}{\epsilon} \nabla_t \cdot (\epsilon \mathbf{e}_t)) - \nabla_t^2 \mathbf{e}_t = (\omega^2 \mu_0 \epsilon - \beta^2)\mathbf{e}_t.$$

This simplifies to the **master equation for waveguide eigenmodes**:

$$(\nabla_t^2 + k_0^2\epsilon_r)\mathbf{e}_t + \nabla_t(\frac{1}{\epsilon_r}\nabla_t\epsilon_r \cdot \mathbf{e}_t) = \beta^2\mathbf{e}_t, \tag{6}$$

where $k_0 = \omega/c$ is the wave number in free space, and $\epsilon_r = \epsilon/\epsilon_0$ is the relative permittivity. This master equation is the fundamental equation for some electromagnetic field simulation sofware, for example, Lumerical finite difference eigenmode (FDE) solver.

Phase relations among field components

For a **lossless waveguide** (real ϵ_r), Eq. 6 contains no explicit imaginary coefficients. Thus, if $\mathbf{e}_t$ is a solution, the real and imaginary parts are also solutions. One may therefore choose $\mathbf{e}_t$ to be **purely real** without loss of generality.

From Eq. 4, a real $\mathbf{e}_t$ implies that e_z is purely imaginary.

The magnetic field components follow similar phase relations. Substituting Eq. 3 into Maxwell's equation Eq. 1b yields

$$\begin{aligned} \nabla_t \times \mathbf{h}_t &= -i\epsilon\omega e_z\hat{z} \\ \hat{z} \times (\nabla_t h_z - i\beta\mathbf{h}_t) &= i\epsilon\omega\mathbf{e}_t. \end{aligned} \tag{7}$$

If $\mathbf{e}_t$ is real and e_z is purely imaginary, it follows from Eq. 7 that $\mathbf{h}_t$ is real and h_z is purely imaginary. Commercial eigenmode solver, such as Lumerical's FDE solver, automatically return modal fields that obey these phase relations: $\mathbf{e}_t, \mathbf{h}_t \in \mathbb{R}$, $e_z, h_z \in i\mathbb{R}$, if the waveguide is lossless.

Linear Algebra Formulation of Waveguide Optics

Eigenvalue problem formulation

The set of all transverse electric fields that can be expressed as linear combinations of the guided-mode profiles forms a vector space, $V = \{\mathbf{v}(x,y) = \sum_k c_k\mathbf{e}_{t,k}, c_k \in \mathcal{C}\}$. Each element of V represents a possible transverse electric-field distribution on a cross-section of the waveguide.

Eq. 6 derived earlier can be interpreted as an **eigenvalue problem** on this vector space. To make this explicit, define the linear operator

$$\Theta(\mathbf{v}) = (\nabla_t^2 + k_0^2\epsilon_r)\mathbf{v} + \nabla_t(\frac{1}{\epsilon_r}\nabla_t\epsilon_r \cdot \mathbf{v}).$$

It's straightforward to verify that Θ is linear: $\Theta(a\mathbf{v}_1 + b\mathbf{v}_2) = a\Theta(\mathbf{v}_1) + b\Theta(\mathbf{v}_2)$, $a, b \in \mathcal{C}$, $\mathbf{v}_1, \mathbf{v}_2 \in V$.

With this operator, the master equation becomes

$$\Theta\mathbf{e}_t = \beta^2\mathbf{e}_t. \tag{8}$$

Thus, finding the modal profile $\mathbf{e}_t$ amounts to identifying the eigenvectors of Θ whose eigenvalues equal β^2. For a lossless waveguide, energy conservation guarantees that β is real; consequently, all eigenvalues of Θ are real. This is reminiscent of quantum mechanics, where one needs to solve for the eigenvectors of a Hermitian Hamiltonian operator with all eigenvalues being real. Each eigenvector is a time-harmonic solution to the Schrodinger's equation and represents a quantum state with well-defined energy $\hbar\omega$, with $\hbar$ the reduced Planck's constant and ω the corresponding eigenvalue.

Remark Since V is constructed as the span of the eigenmodes $\mathbf{e}_{t,k}$ (k is mode index), every vector $\mathbf{v} \in V$ is a linear combination of eigenvectors of Θ. In linear algebra terms, Θ is diagonalizable on V.

Degree of freedom of the EM field in waveguide

A vector $\mathbf{v}(x,y) = \sum_k c_k \mathbf{e}_{t,k} \in V$ represents a transverse electrical field at a given waveguide cross-section. Each transverse modal field $\mathbf{e}_{t,k}$ corresponds to two full EM waves: a forward wave $\mathbf{e}_{t,k}\exp(i\beta_k z)$ and a backward wave $\mathbf{e}_{t,k}\exp(-i\beta_k z)$. Therefore, the full transverse electrical field associated with $\mathbf{v}$ takes the following form:

$$\mathbf{E}_t = \sum_k \mathbf{e}_{t,k}[a_k \exp(i\beta_k z) + b_k \exp(-i\beta_k z)], \tag{9}$$

where a_k and b_k represent the amplitudes of the forward and backward waves, respectively. These amplitudes satisfy $c_k = a_k + b_k$.

This decomposition leads to uniquely defined forward- and backward-propagating fields:

$$\begin{aligned}\mathbf{E}_t^{(f)} &= \sum_k \mathbf{e}_{t,k} c_k \exp(i\beta_k z), \\ \mathbf{E}_t^{(b)} &= \sum_k \mathbf{e}_{t,k} c_k \exp(-i\beta_k z).\end{aligned}$$

Although only the transverse components are shown, these fully determine all field components via Eqs. 9, 4, and 1a. Hence, we have established a one-to-one correspondence between the space V of transverse fields and the spaces of forward- or backward-propagating EM fields, denoted by $V^f = \{\mathbf{E}_t^{(f)}\}$ (or $V^b = \{\mathbf{E}_t^{(b)}\}$). Note that each element of V^f and V^b represents a full-space, not cross-section, EM field in the waveguide.

Remark The complete electromagnetic field is determined uniquely by specifying the electric field on a cross-section. THe transverse field $\mathbf{E}_t$ determines $a_k + b_k$, whereas the longitudinal component determines $a_k - b_k$ through $(a_k + b_k)\nabla_t \cdot (\epsilon \mathbf{e}_t) + i(a_k - b_k)\beta_k \epsilon e_z = 0$, which follows from Eq. 4.

Power flow and inner product

The time-averaged Poynting vector of a harmonic electromagnetic field is: $\mathbf{S} = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*)$. The power transported along the waveguide is obtained by integrating the z-component of $\mathbf{S}$ over a transverse surface S:

$$P = \int_S \mathbf{S} \cdot \hat{z} dA = \frac{1}{2}\int_S \mathrm{Re}(\mathbf{e} \times \mathbf{h}^*) \cdot \hat{z} dA = \frac{1}{4}\int_S [\mathbf{e} \times \mathbf{h}^* + \mathbf{e}^* \times \mathbf{h}] \cdot \hat{z} dA. \tag{10}$$

This motivates the definition of an inner product between two fields in the waveguide $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$:

$$\frac{1}{4}\int_S [\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1] \cdot \hat{z} dA. \tag{11}$$

The definition is physically meaning ful only wen the two fields oscillate at the **same frequency**; otherwise, the time average vanishes. Below we limit our discussion to this case.

Conservation of energy and Lorentz reciprocity relation

For a lossless waveguide, conservation of energy requires that the net time-averaged power flowing out of any closed surface vanish:

$$\frac{1}{4}\oint_S [\mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H}]\cdot\hat{n}dA = 0.$$

In differential form, this condition becomes:

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H}) = 0. \tag{12}$$

A more general identity holds for any two allowed electromagnetic fields $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ at the same frequency:

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2^* + \mathbf{E}_2^*\times\mathbf{H}_1) = 0, \tag{13}$$

This is the celebrated **Lorentz reciprocity relation**. The proof of this relation is given below.

Recall the vector identity

$$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v}). \tag{14}$$

Using Eq. 13 and Eq. 1, we can rewrite the left side of Eq. 12 as:

$$\begin{aligned} &\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2^* + \mathbf{E}_2^*\times\mathbf{H}_1) \\ &= \mathbf{H}_2^*\cdot(\nabla\times\mathbf{E}_1) - \mathbf{E}_1\cdot(\nabla\times\mathbf{H}_2^*) + \mathbf{H}_1\cdot(\nabla\times\mathbf{E}_2^*) - \mathbf{E}_2^*\cdot(\nabla\times\mathbf{H}_1) \\ &= \mathbf{H}_2^*\cdot(i\omega\mu_0\mathbf{H}_1) - \mathbf{E}_1\cdot(i\epsilon\omega\mathbf{E}_2^*) + \mathbf{H}_1\cdot(-i\omega\mu_0\mathbf{H}_2^*) - \mathbf{E}_2^*\cdot(-i\epsilon\omega\mathbf{E}_1) \\ &= 0. \end{aligned}$$

This concludes the proof of the Lorentz reciprocity relation.

Mode orthogonality

Consider two guided modes $(\mathbf{e}_1, \mathbf{h}_1)$ and $(\mathbf{e}_2, \mathbf{h}_2)$ in a **lossless** waveguide, with propagation constants β_1 and $\beta 2$. If $\beta_1 \neq \beta_2$, then these two modes are **orthogonal** in the sense that the inner product (defined Eq. 11) between them is zero. This property of modal fields is referred to as **mode orthogonality**. Below we present a proof of the mode orthogonality.

Starting from the Lorentz reciprocity relation (Eq. 13), applied to the fields $(\mathbf{e}_1 e^{i\beta_1 z}, \mathbf{h}_1 e^{i\beta_1 z})$ and $(\mathbf{e}_2 e^{i\beta_2 z}, \mathbf{h}_2 e^{i\beta_2 z})$, we obtain

$$\nabla\cdot[\mathbf{e}_1\times\mathbf{h}_2^*\exp[i(\beta_1-\beta_2)z] + \mathbf{e}_2^*\times\mathbf{h}_1\exp[i(\beta_1-\beta_2)z]] = 0.$$

Separating transverse and longitudinal derivatives gives

$$[\nabla_t + i(\beta_1 - \beta_2)\hat{z}] \cdot (\mathbf{e}_1 \times \mathbf{h}_2^* + \mathbf{e}_2^* \times \mathbf{h}_1) = 0. \tag{15}$$

Integrating Eq. 15 over the waveguide cross section S yields:

$$\int_S [\nabla_t + i(\beta_1 - \beta_2)\hat{z}] \cdot (\mathbf{e}_1 \times \mathbf{h}_2^* + \mathbf{e}_2^* \times \mathbf{h}_1) dA = 0. \tag{16}$$

Using the 2D divergence theorem, we obtain

$$\int_S \nabla_t \cdot (\mathbf{e}_1 \times \mathbf{h}_2^* + \mathbf{e}_2^* \times \mathbf{h}_1) dA = \oint_C (\mathbf{e}_1 \times \mathbf{h}_2^* + \mathbf{e}_2^* \times \mathbf{h}_1) \cdot \hat{n} dl. \tag{17}$$

where C is a closed contour that encloses the waveguide cross section.

The contour integral in Eq. 17 vanishes for two common lossless waveguides, the dielectric waveguides and metal-boundary waveguides. For a waveguide made of dielectric (i.e., non-conducting) materials, C should be understood as an "infinite" contour. In this case, since the amplitudes of $\mathbf{e}_{1(2)}$ and $\mathbf{h}_{1(2)}$ exponentially decay in the transverse directions outside the waveguide core (high-refractive-index region), the integral evaluates to zero. For a metal-boundary waveguide, C is the contour of the metal boundary. Since the boundary conditions for the modal fields dictate that the transverse components of $\mathbf{e}_{1(2)}$ should be zero, the expression in Eq. 17 again evaluates to zero.

Remark A dielectric waveguide can be viewed as a metal-boundary waveguides whose boundary lies at infinity.

Thus, for a lossless waveguide,

$$\oint_C (\mathbf{e}_1 \times \mathbf{h}_2^* + \mathbf{e}_2^* \times \mathbf{h}_1) \cdot \hat{n} dl = 0. \tag{18}$$

Combining Eqs. 16-18, and using the assumption $\beta_1 \neq \beta_2$, we obtain

$$\frac{1}{4} \int_S (\mathbf{e}_1 \times \mathbf{h}_2^* + \mathbf{e}_2^* \times \mathbf{h}_1) \cdot \hat{z} dA = 0. \tag{19}$$

This is precisely the statement that **two guided modes with different propagation constants are orthogonal.**

Orthonormal basis of EM modes

If a waveguide supports two linear independent modes with the same propagation constant, the modes are said to be **degenerate**. In a waveguide **without** degeneracy, two modes share the same β if and only if they are scalar multiples of each other $\mathbf{e}_2 = c\mathbf{e}_1$, $\mathbf{h}_2 = c\mathbf{h}_1$, $c \neq 0$.

Consider now a nondegenerate waveguide and let $(\mathbf{e}_k, \mathbf{h}_k)$ be the mode associated with propagation constant β_k (k is the mode index). We define a normalized modal pair

$$\mathbf{e}'_k = \frac{\mathbf{e}_k}{\sqrt{\frac{1}{4}|\int_S (\mathbf{e}_k \times \mathbf{h}_k^* + \mathbf{e}_k^* \times \mathbf{h}_k) \cdot \hat{z} dA|}}, \mathbf{h}'_k = \frac{\mathbf{h}_k}{\sqrt{\frac{1}{4}|\int_S (\mathbf{e}_k \times \mathbf{h}_k^* + \mathbf{e}_k^* \times \mathbf{h}_k) \cdot \hat{z} dA|}}. \tag{20}$$

Applying this normalization to every guided mode gives a set $\{(\mathbf{e}'_k, \mathbf{h}'_k)\}$, which forms an **orthonormal basis** for the electromagnetic fields in the waveguide. These modes satisfy the orthonormality relation

$$\frac{1}{4}\int_S (\mathbf{e}_p \times \mathbf{h}_q^* + \mathbf{e}_q^* \times \mathbf{h}_p) \cdot \hat{z} dA = 0 \ if \ \beta_p \neq \beta_q; 1 \ if \ \beta_p = \beta_q > 0; -1 \ if \ \beta_p = \beta_q < 0. \tag{21}$$

Here we have used the fact that β is real for any lossless waveguide.

Remark Even for waveguide *with* degeneracy, it is always possible - by suitable linear combinations of degenerate modes - to construct an orthonormal set satisfying Eq. 21. The only difference is that several orthonormal modes may correspond to the same β.

The power flow has a simple expression when working with the orthonormal basis. For an arbitrary electromagnetic field expanded in the orthonormal modal basis,

$$\mathbf{E} = \sum_k c_k \mathbf{e}'_k \exp(i\beta_k z), \mathbf{H} = \sum_k c_k \mathbf{h}'_k \exp(i\beta_k z),$$

the power carried by the field is:

$$P = \frac{1}{4}\int_S (\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) \cdot \hat{z} dA = \sum_k |c_k|^2 \mathrm{sgn}(\beta_k),$$

where $\mathrm{sgn}(\beta_k)$ equals 1 for $\beta_k > 0$ (forward mode) and -1 for $\beta_k < 0$ (backward mode).

Hilbert space

As discussed previously, each element $\mathbf{v} \in V$ represents a possible transverse electric field distribution on a waveguide cross section. Through the modal expansion, $\mathbf{v}$ uniquely determines a forward-propagating full-space electromagnetic field $(\mathbf{E}, \mathbf{H})$ in the waveguide.

We therefore define the inner product between two elements $\mathbf{v}_1, \mathbf{v}_2 \in V$ as the inner product of their corresponding forward full-space fields $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$:

$$(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{4}\int_S (\mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1) \cdot \hat{z} dA. \tag{22}$$

It's straightforward to check that for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ and all $c_1, c_2 \in \mathcal{C}$, this inner product satisfies $(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_2, \mathbf{v}_1)^*$ (conjugate symmetry) and $(c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \mathbf{v}_3) = c_1(\mathbf{v}_1, \mathbf{v}_3) + c_2(\mathbf{v}_2, \mathbf{v}_3)$ (linearity). Additionally, when $\mathbf{v}_1 \neq 0$, $(\mathbf{v}_1, \mathbf{v}_1)$ is the power flow of a forward EM field and must be a positive number: $(\mathbf{v}_1, \mathbf{v}_1) > 0$ if $\mathbf{v}_1 \neq 0$ (positive-definiteness).

These three properties are precisely the axioms of an inner product on a vector space. Thus, theelectromagnetic inner product induces an inner product on V. Moreover, V is *complete*: any convergent infinite sum of forward-propagating fields corresponds to a physically realizable forward field in the waveguide. Hence, V is a complete inner-product space, i.e., a *Hilbert space*.

Hermitian operator

We now show that the operator Θ introduced earlier is Hermitian on V, meaning that

$$(\Theta \mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, \Theta \mathbf{v}_2), \tag{22}$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$.

Express $\mathbf{v}_1$ and $\mathbf{v}_2$ in the orthonormal eigenbasis $\{\mathbf{e}_{t,k}\}$ of Θ:

$$\mathbf{v}_1 = \sum_k c_{1,k} \mathbf{e}_{t,k}, \mathbf{v}_2 = \sum_k c_{2,k} \mathbf{e}_{t,k},$$

where $\Theta \mathbf{e}_{t,k} = \beta_k^2 \mathbf{e}_{t,k}$, and the eigenvalues β_k^2 are real for a lossless waveguide.

Using the orthonormality of the modes and the reality of the eigenvalues, we obtain

$$(\Theta \mathbf{v}_1, \mathbf{v}_2) = \sum_k c_{1,k} c_{2,k}^* \beta_k^2 = (\mathbf{v}_1, \Theta \mathbf{v}_2).$$

Therefore, Θ is Hermitian on the Hilbert space V.

Power Coupling Efficiency at a Waveguide Transition

Consider two semi-infinite, parallel waveguides joined by a sharp transition. We wish to calculate the power transfer efficiency from the first waveguide to a particular mode of the second waveguide.

Assume the field in the first waveguide contains only forward-propagating components, and neglect any reflection from the interface. At the transition plane, the transverse fields must be continuous and can be expanded in the modal basis of the second waveguide:

$$\mathbf{E}_t = \sum_i a_i \mathbf{e}_{t,i}, \mathbf{H}_t = \sum_i b_i \mathbf{h}_{t,i}, \tag{23}$$

where $\mathbf{E}_t, \mathbf{H}_t$ are the transverse electric and magnetic fields at the interface in the first waveguide, and $\mathbf{e}_{t,i}$ and $\mathbf{h}_{t,i}$ are the transverse fields of the i-th mode of the second waveguide.

We assume that the modal fields of the second waveguide have been chosen to be **real** and **orthonormal**, which is always possible in a lossless structure. In this case, the orthonormality relation becomes

$$\frac{1}{2} \int_S \mathbf{e}_{t,p} \times \mathbf{h}_{t,q} \cdot \hat{z} dA = \delta_{pq}, \tag{24}$$

where δ_{pq} is the Kronecker delta (δ_{pq} equals 1 if $p = q$ and 0 if $p \neq q$).

Combining Eqs. 23 and 24, we obtain the expansion coefficients

$$a_i = \frac{1}{2} \int_S \mathbf{E}_t \times \mathbf{h}_{t,i} \cdot \hat{z} dA, b_i = \frac{1}{2} \int_S \mathbf{e}_{t,i} \times \mathbf{H}_t \cdot \hat{z} dA. \tag{25}$$

The input power carried by the field in the first waveguide is

$$P_{in} = \frac{1}{2}\mathrm{Re}\int_S \mathbf{E}_t \times \mathbf{H}_t^* \cdot \hat{z} dA. \tag{26}$$

The power carried by the i-th mode of the second waveguide is

$$P_i = \frac{1}{2}\mathrm{Re}\int_S a_i b_i^* \mathbf{e}_{t,i} \times \mathbf{h}_{t,i}^* \cdot \hat{z} dA = \frac{1}{2}\mathrm{Re}(a_i b_i^*) \int_S \mathbf{e}_{t,i} \times \mathbf{h}_{t,i} \cdot \hat{z} dA = \mathrm{Re}(a_i b_i^*), \tag{27}$$

where in the last step we used the normalization in Eq. 24.

Subsituting Eq. 25 into Eq. 27 and normalizing by P_{in}, we arrive at

$$\frac{P_i}{P_{in}} = \frac{1}{2}\frac{\mathrm{Re}(\int_S \mathbf{E}_t \times \mathbf{h}_{t,i} \cdot \hat{z} dA \int_S \mathbf{e}_{t,i} \times \mathbf{H}_{t,i}^* \cdot \hat{z} dA)}{\mathrm{Re}\int_S \mathbf{E}_t \times \mathbf{H}_t^* \cdot \hat{z} dA}. \tag{28}$$

The right-hand side of Eq. 28 is commonly referred to as the **mode overlap** between the input field $(\mathbf{E}_t, \mathbf{H}_t)$ and the mode $(\mathbf{e}_{t,i}, \mathbf{h}_{t,i})$ of the second waveguide. Thus, under the assumption of negligible reflection, the **power coupling efficiency** into a given mode is exactly its mode overlap with the incident field.

Remark In Lumerical's FDE solver, the computed mode profiles have real transverse components anad satisfy the orthonormality relation used above. The "mode overlap" reported by the software corresponds precisely to the expansion in Eq. 28.

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